

The Eleven-Dimensional Uplift of Four-Dimensional Supersymmetric RG Flow

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Abstract

The squashed and stretched 7-dimensional internal metric preserving $U(1) \times U(1) \times U(1)_R$ symmetry possesses an Einstein-Kähler 2-fold which is a base manifold of 5-dimensional Sasaki-Einstein $L^{p,q,r}$ space. The r (transverse to the domain wall)-dependence of the two 4-dimensional supergravity fields, that play the role of geometric parameters for squashing and stretching, makes the 11-dimensional Einstein-Maxwell equations consistent not only at the two critical points but also along the whole $\mathcal{N} = 2$ supersymmetric RG flow connecting them. The Ricci tensor of the solution has common feature with the previous three 11-dimensional solutions. The 4-forms preserve only $U(1)_R$ symmetry for other generic parameters of the metric. We find an exact solution to the 11-dimensional Einstein-Maxwell equations corresponding to the lift of the 4-dimensional supersymmetric RG flow.

1 Introduction

The low energy limit of N M2-branes at $\mathbf{C}^4/\mathbf{Z}_k$ singularity is described in the $\mathcal{N} = 6$ $U(N) \times U(N)$ Chern-Simons matter theory with level k in 3-dimensions [1]. For $k = 1, 2$, the enhanced $\mathcal{N} = 8$ supersymmetry is preserved. The matter contents and the superpotential of this theory are the same as the ones in the theory of D3-branes at the conifold [2]. The RG flow of the 3-dimensional theory can be obtained from the 4-dimensional gauged supergravity via AdS/CFT correspondence [3]. The holographic supersymmetric RG flow connecting the maximally supersymmetric point to $\mathcal{N} = 2$ $SU(3) \times U(1)_R$ point has been studied in [4, 5] while those from this maximally supersymmetric point to $\mathcal{N} = 1$ G_2 point has been studied in [5, 6]. The 11-dimensional M-theory uplifts of these have been found in [7, 6] by solving the Einstein-Maxwell equations in 11-dimensions sometime ago.

The mass deformed $U(2) \times U(2)$ Chern-Simons matter theory with level $k = 1, 2$ preserving the above global $SU(3) \times U(1)_R$ symmetry has been studied in [8, 9] while the mass deformation for this theory preserving G_2 symmetry has been described and the further non-supersymmetric RG flow equations preserving two $SO(7)$ symmetries have been discussed in [10]. The holographic RG flow equations connecting G_2 point to $SU(3) \times U(1)_R$ point have been found in [11]. Moreover, the other holographic supersymmetric RG flows have been studied and further developments on the 4-dimensional gauged supergravity have been done in [12]. The spin-2 Kaluza-Klein modes around a warped product of AdS_4 and a seven-ellipsoid which has above global G_2 symmetry are discussed in [13]. The gauge dual with the symmetry of $SU(2) \times SU(2) \times U(1)_R$ for the 11-dimensional lift of $SU(3) \times U(1)_R$ -invariant solution in 4-dimensional supergravity is described in [14]. Recently, the 11-dimensional description preserving $SU(2) \times U(1) \times U(1)_R$ symmetry is found in [15]. They have common $U(1)_R$ factor.

When the 11-dimensional theory from the 4-dimensional gauged supergravity is constructed, the various 11-dimensional solutions will occur even though the flow equations characterized by the 4-dimensional supergravity fields are the same. Since the 4-dimensional flow equations are related to the $\mathcal{N} = 2$ supersymmetry through $U(1)_R$ symmetry, other types of 11-dimensional solutions with common 4-dimensional flow equations will be possible. The invariance of 11-dimensional metric and 4-forms determines each global symmetry. Sometimes the 4-forms restrict to the global symmetry the metric contains and break into the smaller symmetry group. In [7], the two different 11-dimensional solutions where the first has \mathbf{CP}^2 space with $SU(3) \times U(1)_R$ symmetry and the second has $\mathbf{CP}^1 \times \mathbf{CP}^1$ space with $SU(2) \times SU(2) \times U(1)_R$ symmetry are found. Furthermore, the third 11-dimensional solution [15] with a single \mathbf{CP}^1 space with $SU(2) \times U(1) \times U(1)_R$ symmetry is described. The Ricci

tensor for these three solutions with orthonormal frame basis has same value by assuming that the 4-dimensional supergravity fields satisfy the same equations of motion discovered by [4] sometime ago. That is, the same flow equations in 4-dimensions provide three different 11-dimensional solutions to the equations of the motion in 11-dimensional supergravity: $SU(3) \times U(1)_R$ flow corresponding to homogeneous five-sphere, $SU(2) \times SU(2) \times U(1)_R$ flow corresponding to homogeneous $T^{1,1}$ space and $SU(2) \times U(1) \times U(1)_R$ flow corresponding to cohomogeneity one $Y^{p,q}$ space.

In this paper, we construct a new 11-dimensional solution preserving the $U(1)_R$ symmetry. By assuming that the 4-dimensional AdS_4 supergravity fields satisfy the supersymmetric RG flow equations, we find out the correct deformed 7-dimensional internal space possessing the correct global symmetry. By realizing that the 5-dimensional Sasaki-Einstein $Y^{p,q}$ space can be generalized to the 5-dimensional Sasaki-Einstein $L^{p,q,r}$ space [16, 17], we focus on this cohomogeneity two $L^{p,q,r}$ space. When the parameters of the metric satisfy $\alpha = \beta$, the $L^{p,q,r}$ space is nothing but $Y^{p,q}$ space and moreover the isometry of $L^{p,q,r}$ is given by the $U(1) \times U(1) \times U(1)_R$ symmetry. The flow diagrams for four different five-dimensional Sasaki-Einstein spaces ($S^5, T^{1,1}, Y^{p,q}, L^{p,q,r}$) are given in [18].

We will start with the round compactification in terms of $U(1)$ -fibration over the Einstein-Kahler 3-fold, squash this Einstein-Kahler base ellipsoidally, stretch the $U(1)$ fiber, and introduce 3-form tensor gauge potential proportional to the volume form on the base, in the spirit of [7]. Basically the structure of 3-form from the triple wedge product between the orthonormal frames looks similar to the previous three cases. The overall functional dependence on the AdS_4 supergravity fields and the exponential factors corresponding to the unbroken $U(1)$ symmetries can be determined by solving the 11-dimensional Einstein-Maxwell equations directly.

In section 2, starting with the two parts of $L^{p,q,r}$ space metric, $U(1)$ bundle and the Einstein-Kahler 2-fold, we embed them inside of the squashed and stretched 7-dimensional internal space appropriately. Then we determine the full 11-dimensional metric with the correct warp factor. Assuming that the two supergravity fields satisfy the domain wall solutions, we compute the Ricci tensor in this background completely. Surprisingly, the Ricci tensor with orthonormal frame basis has the same value in previous three cases found before. For the 4-form field strengths, we make an ansatz by writing the three pieces: the overall function, the exponential function with $U(1)$'s and the triple wedge product between the orthonormal frames. Finally, we determine the solution for the 11-dimensional Einstein-Maxwell equations.

In section 3, we summarize the results of this paper and present some future directions.

In the Appendix, we present the detailed expressions for the 4-form field strengths.

2 An $\mathcal{N} = 2$ supersymmetric flow in an 11-dimensional theory

Let us describe the 11-dimensional metric. The 3-dimensional metric is given by $\eta_{\mu\nu} = (-, +, +)$, the radial variable r that is the fourth coordinate is transverse to the domain wall, and the scale factor $A(r)$ in 3-dimensional metric behaves linearly in r at UV and IR regions. The 4-dimensional spacetime metric in the 11-dimensions contains a warp factor $\Delta(r, \mu)$ that depends on the μ which is the fifth coordinate as well as the r . The 7-dimensional internal metric depends on the 4-dimensional supergravity fields (ρ, χ) and the warp factor in the full 11-dimensions. Explicitly, we have the 11-dimensional metric as follows [7, 19]:

$$ds_{11}^2 = \Delta(r, \mu)^{-1} (dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu) + L^2 \Delta(r, \mu)^{\frac{1}{2}} ds_7^2(\rho, \chi), \quad (2.1)$$

where L is a radius of round seven-sphere.

In 4-dimensions, there are two critical points, $\mathcal{N} = 8$ $SO(8)$ critical point and $\mathcal{N} = 2$ $SU(3) \times U(1)_R$ critical point. At these points, the derivatives of superpotential $W(\rho, \chi)$ with respect to the supergravity fields vanish. Let us recall that the supergravity fields (ρ, χ) and the scale function A appearing in the metric (2.1) satisfy the supersymmetric RG flow equations [4]:

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{1}{8L\rho} [(\cosh(2\chi) + 1) + \rho^8 (\cosh(2\chi) - 3)], \\ \frac{d\chi}{dr} &= \frac{1}{2L\rho^2} (\rho^8 - 3) \sinh(2\chi), \\ \frac{dA}{dr} &= \frac{1}{4L\rho^2} [3(\cosh(2\chi) + 1) - \rho^8 (\cosh(2\chi) - 3)]. \end{aligned} \quad (2.2)$$

As explained in the introduction, let us consider the 4-dimensional Einstein-Kähler 2-fold which lives in the five-dimensional $L^{p,q,r}$ space [16, 17]. The one-form containing the $U(1)$ bundle over this Einstein-Kähler 2-fold consists of two parts as follows [7, 19]:

$$\omega = \frac{1}{2} \sin(2\mu) [-\rho(r)^{-4} d\gamma + \rho(r)^4 (u, Jdu)], \quad (2.3)$$

where the 8-dimensional vector $u = (u^1, \dots, u^6, 0, 0)$ parametrizes a unit five-sphere, γ is an 11-th coordinate and J is the Kähler form that has $J_{12} = J_{34} = J_{56} = J_{78} = 1$ explicitly.

One needs to know (u, Jdu) in (2.3) corresponding to the $U(1)$ bundle over the Einstein-Kähler 2-fold. Let us recall the metric for the 5-dimensional Sasaki-Einstein space $L^{p,q,r}$ used in [17]

$$ds_{L^{p,q,r}}^2 = ds_{EK(2)}^2 + [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi]^2, \quad (2.4)$$

with the Einstein-Kahler 2-fold ¹

$$ds_{EK(2)}^2 = \frac{(\eta - \xi)}{2G(\eta)} d\eta^2 + \frac{(\eta - \xi)}{2F(\xi)} d\xi^2 + \frac{2F(\xi)}{(\eta - \xi)} (d\phi + \eta d\psi)^2 + \frac{2G(\eta)}{(\eta - \xi)} (d\phi + \xi d\psi)^2 \quad (2.5)$$

where the ξ and η -dependent cubic functions with parameters α, β, μ_1 are given by

$$F(\xi) \equiv 2\xi(\alpha - \xi)(\alpha - \beta - \xi), \quad G(\eta) \equiv -2\eta(\alpha - \eta)(\alpha - \beta - \eta) - 2\mu_1. \quad (2.6)$$

It is obvious that the form in the last term of (2.4) provides the Kahler 2-form and satisfies

$$d[-2(\xi + \eta) d\phi - 2\xi\eta d\psi] = -2[d\xi \wedge (d\phi + \eta d\psi) + d\eta \wedge (d\phi + \xi d\psi)]. \quad (2.7)$$

Therefore, one identifies (u, Jdu) with the $U(1)$ bundle over the Einstein-Kahler 2-fold as follows:

$$(u, Jdu) = d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi. \quad (2.8)$$

What about the $U(1)$ Hopf fiber (x, Jdx) on \mathbf{CP}^3 where $x = (x^1, \dots, x^8)$ is a vector on 8-dimensional space in terms of (u, Jdu) ? This $U(1)$ Hopf fiber becomes [7, 19]

$$(x, Jdx) = \cos^2 \mu (u, Jdu) + \sin^2 \mu d\gamma. \quad (2.9)$$

The 7-dimensional internal space metric $ds_7^2(\rho, \chi)$ appearing in the metric (2.1) can be written as ²

$$ds_7^2 = \rho(r)^{-4} \Xi^2 d\mu^2 + \rho(r)^2 \cos^2 \mu ds_{EK(2)}^2 + \Xi^{-2} \omega^2 + \Xi^{-2} \cosh^2 \chi(r) (x, Jdx)^2. \quad (2.10)$$

By substituting the metric (2.5) for the Einstein-Kahler 2-fold into the second term of (2.10), plugging the 1-form (2.3) together with (2.8) into the third term of (2.10) and substituting the $U(1)$ Hopf fiber (2.9) into the last term of (2.10), finally one obtains the final 7-dimensional internal metric preserving $U(1) \times U(1) \times U(1)_R$ symmetry as follows:

$$\begin{aligned} ds_7^2(\rho, \chi) = & \rho(r)^{-4} \Xi(r, \mu)^2 d\mu^2 \\ & + \rho(r)^2 \cos^2 \mu \left[\frac{(\eta - \xi)}{2G(\eta)} d\eta^2 + \frac{(\eta - \xi)}{2F(\xi)} d\xi^2 + \frac{2F(\xi)}{(\eta - \xi)} (d\phi + \eta d\psi)^2 + \frac{2G(\eta)}{(\eta - \xi)} (d\phi + \xi d\psi)^2 \right] \\ & + \Xi(r, \mu)^{-2} \frac{1}{4} \sin^2(2\mu) \left(-\rho(r)^{-4} d\gamma + \rho(r)^4 [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi] \right)^2 \\ & + \Xi(r, \mu)^{-2} \cosh^2 \chi(r) \left(\sin^2 \mu d\gamma + \cos^2 \mu [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi] \right)^2. \end{aligned} \quad (2.11)$$

¹One can also consider the metric by [16] but the trigonometric functions on the angle θ appear in the metric and this makes the Ricci tensor be complicated expressions. However, the parametrization of [17] we use in this paper can make the metric (2.4) take the form of [16]. So we take the convention of [17].

²The parameter ξ in [7] is replaced by a capital letter Ξ in order not to confuse with the coordinate ξ in (2.5).

Here one has

$$\Xi(r, \mu) = \frac{\sqrt{X(r, \mu)}}{\rho(r)}, \quad X(r, \mu) \equiv \cos^2 \mu + \rho(r)^8 \sin^2 \mu. \quad (2.12)$$

The nontrivial squashing characterized by $\rho(r)$ deforms the metric on the \mathbf{CP}^3 and moreover rescales the Hopf fiber which appears in the last line of (2.11). The stretching is characterized by $\chi(r)$. There exists $U(1) \times U(1)$ symmetry from the structure of Einstein-Kahler 2-fold in $ds_{EK(2)}^2$. These two $U(1)$ symmetries are generated by the 8-th coordinate ϕ and 9-th coordinate ψ . The combined two $U(1)$ symmetries by the 10-th coordinate τ and 11-th coordinate γ will provide a single $U(1)_R$ symmetry relevant to the $\mathcal{N} = 2$ supersymmetry later.

For $\mu = 0$, the 7-dimensional metric (2.11) reduces to the following metric on moduli space for the M2-brane probe

$$\rho(r)^2 ds_{L^{p,q,r}}^2 + \rho(r)^2 \sinh^2 \chi(r) [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi]^2, \quad (2.13)$$

where the metric for $L^{p,q,r}$ is given by (2.4). The function $\sinh^2 \chi(r)$ in (2.13) plays the role of a stretching of the $U(1)$ -fiber. Then for this particular coordinate $\mu = 0$ there exists a stretched $L^{p,q,r}$ space.

One obtains the following set of orthonormal frames for the 11-dimensional metric (2.1) as follows:

$$\begin{aligned} e^1 &= -\Delta(r, \mu)^{-\frac{1}{2}} e^{A(r)} dx^1, & e^2 &= \Delta(r, \mu)^{-\frac{1}{2}} e^{A(r)} dx^2, & e^3 &= \Delta(r, \mu)^{-\frac{1}{2}} e^{A(r)} dx^3, \\ e^4 &= \Delta(r, \mu)^{-\frac{1}{2}} dr, & e^5 &= L \Delta(r, \mu)^{\frac{1}{4}} \frac{\sqrt{X(r, \mu)}}{\rho(r)^3} d\mu, \\ e^6 &= L \Delta(r, \mu)^{\frac{1}{4}} \rho(r) \cos \mu \sqrt{\frac{\eta - \xi}{2G(\eta)}} d\eta, \\ e^7 &= L \Delta(r, \mu)^{\frac{1}{4}} \rho(r) \cos \mu \sqrt{\frac{\eta - \xi}{2F(\xi)}} d\xi, \\ e^8 &= L \Delta(r, \mu)^{\frac{1}{4}} \rho(r) \cos \mu \sqrt{\frac{2F(\xi)}{\eta - \xi}} [d\phi + \eta d\psi], \\ e^9 &= L \Delta(r, \mu)^{\frac{1}{4}} \rho(r) \cos \mu \sqrt{\frac{2G(\eta)}{\eta - \xi}} [d\phi + \xi d\psi], \\ e^{10} &= L \Delta(r, \mu)^{\frac{1}{4}} \frac{\rho(r)}{\sqrt{X(r, \mu)}} \frac{1}{2} \sin(2\mu) (-\rho(r)^{-4} d\gamma + \rho(r)^4 [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi]), \\ e^{11} &= L \Delta(r, \mu)^{\frac{1}{4}} \frac{\rho(r) \cosh \chi(r)}{\sqrt{X(r, \mu)}} (\sin^2 \mu d\gamma + \cos^2 \mu [d\tau - 2(\xi + \eta) d\phi - 2\xi\eta d\psi]), \end{aligned} \quad (2.14)$$

where the warp factor is

$$\Delta(r, \mu) = \frac{\rho(r)^{\frac{4}{3}}}{X(r, \mu)^{\frac{2}{3}} \cosh^{\frac{4}{3}} \chi(r)}. \quad (2.15)$$

The Einstein-Maxwell equations are given by [20, 21]

$$\begin{aligned} R_M^N &= \frac{1}{3} F_{MPQR} F^{NPQR} - \frac{1}{36} \delta_M^N F_{PQRS} F^{PQRS}, \\ \nabla_M F^{MNPQ} &= -\frac{1}{576} E \epsilon^{NPQRSTUUVWXY} F_{RSTU} F_{VWXY}, \end{aligned} \quad (2.16)$$

where the covariant derivative ∇_M on F^{MNPQ} is given by $E^{-1} \partial_M (E F^{MNPQ})$ together with elfbein determinant $E \equiv \sqrt{-g_{11}}$. The epsilon tensor $\epsilon_{NPQRSTUUVWXY}$ with lower indices is purely numerical. All the indices in (2.16) are based on the coordinate basis. For given 11-dimensional metric (2.1) together with (2.15) and (2.11) or (2.14), the next step is to find the solution for (2.16).

Let us describe the 11-dimensional solution at two critical points and after that along the whole RG flow connecting them.

- At the UV fixed point

At this critical point

$$\rho(r) = 1, \quad \chi(r) = 0, \quad (2.17)$$

one recovers the maximally supersymmetric $\mathcal{N} = 8$ $AdS_4 \times \mathbf{S}^7$ solution [21] and the Ricci tensor has the form

$$R_M^N = \frac{6}{L^2} \text{diag}(-2, -2, -2, -2, 1, 1, 1, 1, 1, 1, 1).$$

The 3-form gauge field with 3-dimensional M2-brane indices is defined by [7]

$$A^{(3)} = \frac{1}{2} e^{\frac{6r}{L}} dx^1 \wedge dx^2 \wedge dx^3. \quad (2.18)$$

At the UV end, the scale function $A(r)$ behaves as $\frac{2}{L}r$ and one obtains the only nonzero component for the 4-form as $F_{1234} = -\frac{18}{L}$ [22].

- At the IR fixed point

At this critical point

$$\rho(r) = 3^{\frac{1}{8}}, \quad \chi(r) = \frac{1}{2} \cosh^{-1} 2, \quad (2.19)$$

the function $A(r)$ behaves as $\frac{3^{\frac{3}{4}}}{L}r$ ($\hat{L} \equiv 3^{-\frac{3}{4}}L$), then one writes down the 3-form gauge field as follows [7]:

$$A^{(3)} = \frac{3^{\frac{3}{4}}}{4} e^{\frac{3r}{L}} dx^1 \wedge dx^2 \wedge dx^3 + C^{(3)} + (C^{(3)})^*. \quad (2.20)$$

Since the Kahler form in (2.7) contains $e^6 \wedge e^9$ and $e^7 \wedge e^8$, this leads to the natural basis of the one-forms and the \mathbf{CP}^3 factor for $\rho = 1$ and $\chi = 0$ (2.17) has also e^5 and e^{10} which can be combined together. In fact, we find

$$C^{(3)} = -\frac{1}{4} \sinh \chi(r) e^{-i[4(\beta-2\alpha)\phi+2\alpha(\beta-\alpha)\psi+3\tau+\gamma]} (e^5 + ie^{10}) \wedge (e^6 + ie^9) \wedge (e^7 + ie^8). \quad (2.21)$$

Although the structure of triple wedge product in (2.21) between the orthonormal basis looks very similar to the previous constructions with $SU(3) \times U(1)_R$ symmetry [7], $SU(2) \times SU(2) \times U(1)_R$ symmetry [7] or $SU(2) \times U(1) \times U(1)_R$ symmetry [15], the functional behavior of the exponential function in 3-form behave differently.

The Ricci tensor has only two nonvanishing off-diagonal components: R_{10}^{11} and R_{11}^{10} . It turns out the Ricci tensor is identical to the one with $SU(3) \times U(1)_R$ symmetry [7], the one with $SU(2) \times SU(2) \times U(1)_R$ symmetry [7, 14] or the one with $SU(2) \times U(1) \times U(1)_R$ symmetry [15]. That is, the Ricci tensor for four cases has same value (in the frame basis) at the IR critical point (2.19). They are given by [14, 15]

$$\begin{aligned} R_1^1 &= -\frac{(55 - 32 \cos 2\mu + 3 \cos 4\mu)}{3 \cdot 2^{\frac{1}{3}} \sqrt{3} \hat{L}^2 (2 - \cos 2\mu)^{\frac{8}{3}}} = R_2^2 = R_3^3 = R_4^4 = -2R_6^6 = -2R_7^7 = -2R_8^8 = -2R_9^9, \\ R_5^5 &= \frac{(29 - 16 \cos 2\mu)}{3 \cdot 2^{\frac{1}{3}} \sqrt{3} \hat{L}^2 (2 - \cos 2\mu)^{\frac{8}{3}}} = R_{10}^{10}, \quad R_{10}^{11} = -\frac{2 \cdot 2^{\frac{1}{6}} \sin 2\mu}{\sqrt{3} \hat{L}^2 (2 - \cos 2\mu)^{\frac{5}{3}}} = R_{11}^{10}, \\ R_{11}^{11} &= \frac{(80 - 64 \cos 2\mu + 9 \cos 4\mu)}{3 \cdot 2^{\frac{1}{3}} \sqrt{3} \hat{L}^2 (2 - \cos 2\mu)^{\frac{8}{3}}}. \end{aligned} \quad (2.22)$$

They depend on only the fifth coordinate μ . One transforms the Einstein equation with coordinate basis into the one with frame basis via (2.14). By comparing the (10, 11) component of Einstein equation, the coefficients for the angles τ and γ which are equal to -3 and -1 and the overall coefficient of 3-form that is $-\frac{1}{4}$ are completely fixed. Moreover, the (10, 9) component of right hand side of Einstein equation is nonzero but the corresponding R_{10}^9 from (2.22) vanishes. This implies that the coefficient of ϕ should be $8\alpha - 4\beta$ and the coefficient of ψ should be $2\alpha^2 - 2\alpha\beta$ in the exponent of 3-form (2.21). Then there exists a $U(1)$ symmetry generated by the angle ϕ or ψ with particular conditions on α and β . Either $(2\alpha - \beta) = 0$ or $\alpha(\alpha - \beta) = 0$.

The internal part of $F^{(4)}$ can be written as $dC^{(3)} + d(C^{(3)})^*$ with (2.21) and the anti-symmetric tensor fields can be obtained from $F^{(4)} = dA^{(3)}$ with (2.20). It turns out that the antisymmetric field strengths have the following nonzero components in the orthonormal

frame basis:

$$\begin{aligned}
F_{1234} &= -\frac{3 \cdot 2^{\frac{1}{3}} \cdot 3^{\frac{3}{4}}}{\hat{L}(2 - \cos 2\mu)^{\frac{4}{3}}}, \\
F_{579\,10} + i F_{567\,10} &= \frac{2^{\frac{1}{3}} \cdot 3^{\frac{3}{4}} \sin 2\mu}{\hat{L}(2 - \cos 2\mu)^{\frac{4}{3}}} e^{i\delta} = -F_{568\,10} + i F_{589\,10}, \\
F_{579\,11} + i F_{567\,11} &= -\frac{2^{\frac{5}{6}} \cdot 3^{\frac{3}{4}}}{\hat{L}(2 - \cos 2\mu)^{\frac{1}{3}}} e^{i\delta} = -F_{568\,11} + i F_{589\,11}, \\
&= -F_{67\,10\,11} + i F_{79\,10\,11} = -F_{89\,10\,11} - i F_{68\,10\,11},
\end{aligned} \tag{2.23}$$

where we introduce the exponent appearing in (2.21) as a single variable

$$\delta \equiv 4(\beta - 2\alpha)\phi + 2\alpha(\beta - \alpha)\psi + 3\tau + \gamma. \tag{2.24}$$

The angle-dependences for ϕ, ψ, τ and γ appear via (2.24). One can make the four $U(1)$ symmetries generated by these angles which preserve the δ . These 4-forms break the $U(1) \times U(1) \times U(1) \times U(1)$ into $U(1)_R$. After substituting (2.23) into the right hand side of Einstein equation (2.16) with frame basis (2.14) one reproduces the one for $SU(3) \times U(1)_R$ symmetry case [7], $SU(2) \times SU(2) \times U(1)_R$, or $SU(2) \times U(1) \times U(1)_R$ [15] exactly. This feature is also expected because the Ricci tensor for four independent cases is identical to each other. That is, the 4-forms themselves are different from each other but their quadratic combinations appearing in the right hand side of Einstein equation are the same. Note that the 4-form given in (2.23) looks very similar to the one of $SU(2) \times SU(2) \times U(1)_R$ symmetry case [7, 14] or $SU(2) \times U(1) \times U(1)_R$ symmetry case [15]: same independent components(up to signs).

- Along the the RG flow

Now let us consider the whole RG flow. For solutions with varying scalars, the ansatz for the 4-form field strength will be more complicated. We apply the correct ansatz for the 11-dimensional 3-form gauge field by acquiring the r -dependence of the 4-dimensional supergravity scalars and derive the 11-dimensional Einstein-Maxwell equations corresponding to the $U(1)_R$ -invariant RG flow.

Let us take the 3-form ansatz as follows:

$$A^{(3)} = \widetilde{W}(r, \mu) e^{3A(r)} dx^1 \wedge dx^2 \wedge dx^3 + C^{(3)} + (C^{(3)})^*, \tag{2.25}$$

where $C^{(3)}$ is given by (2.21) as before. One puts an arbitrary function $f(\rho, \chi)$ in front of this 3-form at the beginning. One obtains the Ricci tensor from the 11-dimensional metric (2.1) when the supergravity fields (ρ, χ) vary with respect to the r -coordinate. They are exactly the same as the one in the Appendix A of [15]. The (10, 11) component of Einstein equation

determines the function $f(\rho, \chi)$. One obtains $vf(v) + (1 - v^2)f'(v) = 0$ where $v \equiv \cosh \chi$. This implies that the solution $f(v)$ is exactly the same as $\sinh \chi$ which appears in (2.21).

One determines the exact form for the geometric superpotential introduced in (2.25). Let us consider (4, 4), (4, 5)- and (5, 5)-components of the right hand side of Einstein equation. By eliminating $(\partial_r \widetilde{W})^2$ from (4, 4)- and (5, 5)-components, one obtains $\partial_\mu \widetilde{W}(r, \mu)$. By integrating this with respect to the μ coordinate, one gets $\widetilde{W}(r, \mu)$ with undetermined function for the r . By making the correct ansatz for this function, one can determine it completely from the (4, 5) component of Einstein equation. Actually, the (4, 4), (4, 5)- and (5, 5)-components of the right hand side of Einstein equation are exactly the same as the ones in [15]. Therefore, one obtains the final form for the geometric superpotential as follows:

$$\widetilde{W}(r, \mu) = \frac{1}{4\rho(r)^2} [(\cosh 2\chi(r) + 1) \cos^2 \mu - \rho(r)^8 (\cosh 2\chi(r) - 3) \sin^2 \mu], \quad (2.26)$$

which is exactly the same as the one [7] found in other three cases. Note that this reproduces the UV value in (2.18) or the IR value in (2.20).

Comparing with the previous 4-form fields at the IR fixed point, the mixed 4-form fields $F_{\mu\nu\rho 5}$, F_{4mnp} and F_{45mn} where $\mu, \nu, \rho = 1, 2, 3$ and $m, n, p = 6, 7, \dots, 11$ are new if we look at the (A.1). We also present the 4-forms in the coordinate basis in Appendix B. For the checking of the remaining Maxwell equation (2.16), one needs to know the elfbein determinant $E = \sqrt{-g_{11}}$ and it turns out that it is given by

$$E = 243 \cdot 3^{\frac{1}{4}} e^{3A(r)} \hat{L}^7 \rho(r)^{-\frac{4}{3}} \cosh^{\frac{4}{3}} \chi(r) (\eta - \xi) \cos^5 \mu \sin \mu (\cos^2 \mu + \rho(r)^8 \sin^2 \mu)^{\frac{2}{3}}.$$

Moreover, the determinant of inverse metric is $g_{11}^{-1} = \epsilon^{1234567891011} = -E^{-2}$. We have checked that all of the Maxwell equations of motion are indeed satisfied.

Thus we have constructed that the solutions (2.25), (2.21), and (2.26) consist of an exact solution to the 11-dimensional supergravity characterized by bosonic field equations (2.16), provided that the deformation parameters (ρ, χ) of the 7-dimensional internal space and the domain wall amplitude A develop in the AdS_4 radial direction along the RG flow (2.2).

For $\alpha = \beta$ (after we go to the metric by [16]), then the metric of (2.4) leads to the standard metric of $Y^{p,q} = L^{p-q, p+q, p}$ space. For $p = q = r = 1$, the metric provides the homogeneous $T^{1,1}$ space. For $\mu_1 = 0$, the metric becomes the round five-sphere metric [16]. In the 11-dimensional view point, the four independent RG flows characterized by

$$\begin{aligned} \mathbf{S}^5 - \text{flow} & : SU(3) \times U(1)_R, \\ T^{1,1} - \text{flow} & : SU(2) \times SU(2) \times U(1)_R, \\ Y^{p,q} - \text{flow} & : SU(2) \times U(1) \times U(1)_R, \\ L^{p,q,r} - \text{flow} & : U(1) \times U(1)_R, \quad U(1)_R, \end{aligned} \quad (2.27)$$

arrive at the IR fixed point at which they have common Ricci tensor given in the Appendix A of [15]. Depending on their global symmetry, the internal 3-forms, in each case, have the right structures in the exponential function with common $\sinh \chi$ -dependence. However, the 3-form in the M2-brane world-volume directions with the same geometric superpotential (2.26) is common to four different solutions (2.27). Although the 4-forms are different from each other completely, the squares of these 4-forms appearing in the right hand side of Einstein equation (2.16) give rise to the same expressions.

3 Conclusions and outlook

The solutions, characterized by (2.21), (2.25) and (2.26) together with (2.14), for 11-dimensional Einstein-Maxwell equations corresponding to the $\mathcal{N} = 2$ $U(1)_R$ -invariant RG flow in the 4-dimensional gauged supergravity are found. More explicitly, the Ricci tensor is given by the Appendix A of [15] and the 4-forms are given by the Appendix B of this paper. These two quantities are the basic objects in the 11-dimensional Einstein-Maxwell equations (2.16). The 4-forms with upper indices can be obtained from those with lower indices given in the Appendix B via the 11-dimensional metric (2.14). Note that the $U(1) \times U(1) \times U(1)_R$ symmetry of 11-dimensional metric breaks into $U(1)_R$ in the presence of 4-form field strengths for general parameters α and β . One can interpret the AdS_4 supergravity fields as the geometric parameters for the 7-dimensional internal space and as long as the r -dependence of these fields is controlled by the supersymmetric RG flow equations, the exact solution for the 11-dimensional field equations is determined. Therefore the $U(1)_R$ -invariant holographic RG flow is lifted to an $\mathcal{N} = 2$ M2-brane flow in M-theory.

It is an open problem to find out what is corresponding dual gauge theory for the 11-dimensional background we have described in the context of AdS/CFT. In [23], the higher dimensional analog of the 5-dimensional $Y^{p,q}$ space was found. Then the partial resolution of the 7-dimensional space might be a candidate for the dual gauge theory, along the line of [24]. It would be interesting to study the other possibility where there exists a bigger $SU(3) \times U(1) \times U(1)_R$ symmetry for the 11-dimensional lift of the same 4-dimensional RG flow equations. The 11-dimensional lift of $\mathcal{N} = 1$ G_2 invariant theory was found in [6]. It is an open problem whether one can embed the appropriate Einstein-Kähler 2-fold inside of six-sphere or whether one can replace other Einstein-Kähler 3-fold. As mentioned in the introduction, due to the flow from G_2 point to $SU(3) \times U(1)_R$ point in 4-dimensions, one expects that there should be 11-dimensional uplifts of 4-dimensional flow equations satisfied in the G_2 point corresponding to (2.27).

Appendix A The 4-form field strength in frame basis

One can read off the 4-forms from (2.21), (2.25) and (2.26) and they are given in the frame basis as follows:

$$\begin{aligned}
F_{1234} &= \frac{3^{\frac{1}{4}} [c_\mu^2 (-5 + \cosh 2\chi) + 2\rho^8 (-2 + c_{2\mu} + s_\mu^2 \rho^8 \sinh^2 \chi)]}{\hat{L} \rho^{\frac{4}{3}} \cosh^{\frac{2}{3}} \chi (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{4}{3}}}, \\
F_{579\,10} + i F_{567\,10} &= \frac{3^{\frac{1}{4}} \rho^{\frac{8}{3}} (-1 + \rho^8) \sinh \chi}{\hat{L} \operatorname{sech}^{\frac{1}{3}} \chi (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{4}{3}}} s_{2\mu} e^{i\delta} = -F_{568\,10} + i F_{589\,10}, \\
F_{579\,11} + i F_{567\,11} &= -\frac{3^{\frac{1}{4}} (3 + \rho^8) \sinh \chi}{\hat{L} \rho^{\frac{4}{3}} \cosh^{\frac{2}{3}} \chi (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{1}{3}}} e^{i\delta} = -F_{568\,11} + i F_{589\,11} \\
&= -F_{67\,10\,11} + i F_{79\,10\,11} = -F_{89\,10\,11} - i F_{68\,10\,11}, \\
F_{1235} &= \frac{3^{\frac{1}{4}} \rho^{\frac{8}{3}} [1 + \cosh 2\chi + \rho^8 (-3 + \cosh 2\chi)]}{\hat{L} \cosh^{\frac{5}{3}} \chi (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{4}{3}}} s_\mu c_\mu, \\
F_{4567} + i F_{4568} &= -\frac{3^{\frac{1}{4}} (-3 + \rho^8) \sinh 2\chi}{2\hat{L} \rho^{\frac{4}{3}} \cosh^{\frac{5}{3}} \chi (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{1}{3}}} e^{i\delta} = F_{4589} - i F_{4579} \\
&= -F_{468\,10} + i F_{467\,10} = F_{479\,10} + i F_{489\,10}, \\
F_{479\,11} + i F_{467\,11} &= -\frac{3^{\frac{1}{4}} \rho^{\frac{8}{3}} \operatorname{sech}^{\frac{5}{3}} \chi \sinh \chi [1 + \cosh 2\chi + \rho^8 (-3 + \cosh 2\chi)]}{2\hat{L} (c_\mu^2 + \rho^8 s_\mu^2)^{\frac{4}{3}}} s_{2\mu} e^{i\delta} \\
&= -F_{468\,11} + i F_{489\,11}. \tag{A.1}
\end{aligned}$$

For simplicity, we ignored the r -dependence on ρ and χ in the right hand side of (A.1). At the IR critical point, the only first half of these survives, which is consistent with (2.23).

Appendix B The 4-form field strength in coordinate basis

For convenience, let us present the 4-forms in coordinate basis. First of all, the 3-form gauge field with 3-dimensional M2-brane indices appearing in (2.25) provides the following two 4-forms. One is given by

$$F_{1234} = \frac{3^{\frac{1}{4}}}{2\hat{L}\rho^4} e^{3A} [2c_\mu^2 \cosh^2 \chi (-5 + \cosh 2\chi) + 4\rho^8 (-2 + c_{2\mu}) \cosh^2 \chi + \rho^{16} s_\mu^2 \sinh^2 2\chi],$$

where $r \equiv x^4$ and the other is

$$F_{1235} = \frac{3}{2\rho^2} e^{3A} s_{2\mu} [1 + \cosh 2\chi + \rho^8 (-3 + \cosh 2\chi)],$$

which vanishes at IR point and $\mu \equiv x^5$. The six (45mn)-components, in unusual notation, are given by

$$\frac{\sqrt{FG}(F_{4567}, F_{4568}, F_{4569}, F_{4578}, F_{4579}, F_{4589})}{9\sqrt{3}\hat{L}^2 c_\mu^2 \tanh \chi \rho^{-2} (-3 + \rho^8)} = \left[-\frac{1}{2}(\eta - \xi)c_\delta, -Fs_\delta, -\eta Fs_\delta, Gs_\delta, \xi Gs_\delta, 2FGc_\delta \right],$$

which vanish at the IR-critical point, and the sixteen (4mnp)-components are given by

$$\begin{aligned} & \frac{\sqrt{FG}(F_{4678}, F_{4679}, F_{46710}, F_{4689}, F_{46810}, F_{46910}, F_{4789}, F_{47810}, F_{47910}, F_{48910})}{9\sqrt{3}\hat{L}^2 c_\mu^3 s_\mu \tanh \chi X^{-2} \rho^6 [c_\mu^2(-2 + \cosh 2\chi) + (-2 + \cosh 2\chi - 2 \cosh^2 \chi s_\mu^2)\rho^8 + s_\mu^2 \rho^{16}]} \\ &= \left[(\eta^2 - \xi^2)s_\delta, \eta\xi(\eta - \xi)s_\delta, -\frac{1}{2}(\eta - \xi)s_\delta, 2\eta^2 Fc_\delta, Fc_\delta, \eta Fc_\delta, -2\xi^2 Gc_\delta, -Gc_\delta, -\xi Gc_\delta, 2FGs_\delta \right] \end{aligned}$$

and

$$\begin{aligned} & \frac{\sqrt{FG}(F_{46711}, F_{46811}, F_{46911}, F_{47811}, F_{47911}, F_{48911})}{9\sqrt{3}\hat{L}^2 c_\mu^3 s_\mu \tanh \chi X^{-2} \rho^{-2} [3c_\mu^2 + (1 - 2c_{2\mu} + 2 \cosh^2 \chi s_\mu^2)\rho^8 + (-4 + \cosh 2\chi) s_\mu^2 \rho^{16}]} \\ &= \left[-\frac{1}{2}(\eta - \xi)s_\delta, Fc_\delta, \eta Fc_\delta, -Gc_\delta, -\xi Gc_\delta, 2FGs_\delta \right], \end{aligned}$$

where the functions F, G and X are given by (2.6) and (2.12). These (4mnp)-components vanish at the IR-critical point also. The sixteen (5mnp)-components are

$$\begin{aligned} & \frac{\sqrt{FG}(F_{5678}, F_{5679}, F_{56710}, F_{5689}, F_{56810}, F_{56910}, F_{5789}, F_{57810}, F_{57910}, F_{58910})}{27 \cdot 3^{\frac{1}{4}} \hat{L}^3 c_\mu^4 \tanh \chi X^{-2} [3c_\mu^2 + (3 - 2c_{2\mu})\rho^8 - s_\mu^2 \rho^{16}]} \\ &= \left[(\eta^2 - \xi^2)s_\delta, \eta\xi(\eta - \xi)s_\delta, -\frac{1}{2}(\eta - \xi)s_\delta, 2\eta^2 Fc_\delta, Fc_\delta, \eta Fc_\delta, -2\xi^2 Gc_\delta, -Gc_\delta, -\xi Gc_\delta, 2FGs_\delta \right], \end{aligned}$$

and

$$\frac{4\sqrt{FG}(F_{56711}, F_{56811}, F_{56911}, F_{57811}, F_{57911}, F_{58911})}{27 \cdot 3^{\frac{1}{4}} \hat{L}^3 s_\mu^2 \tanh \chi X^{-2} (c_\mu^2 + 3\rho^8 + s_\mu^2 \rho^{16})} = \left[-\frac{1}{2}(\eta - \xi)s_\delta, Fc_\delta, \eta Fc_\delta, -Gc_\delta, -\xi Gc_\delta, 2FGs_\delta \right].$$

Finally, the ten (mnpq)-components are given by

$$\begin{aligned} & \frac{\sqrt{FG}(F_{67811}, F_{67911}, F_{671011}, F_{68911}, F_{681011}, F_{691011}, F_{78911}, F_{781011}, F_{791011}, F_{891011})}{27 \cdot 3^{\frac{1}{4}} \hat{L}^3 c_\mu^3 s_\mu \tanh \chi X^{-1} (3 + \rho^8)} = \\ & \left[-(\eta^2 - \xi^2)c_\delta, -\eta\xi(\eta - \xi)c_\delta, \frac{1}{2}(\eta - \xi)c_\delta, 2\eta^2 Fs_\delta, Fs_\delta, \eta Fs_\delta, -2\xi^2 Gs_\delta, -Gs_\delta, -\xi Gs_\delta, -2FGc_\delta \right]. \end{aligned}$$

Compared with the ones in Appendix A, the extra $F_{4569}, F_{4578}, F_{469m}, F_{478m}, F_{569m}, F_{578m}, F_{69m11}$ and F_{78m11} components in coordinate basis are new. The $F_{4589}, F_{48910}, F_{48911}, F_{58910}, F_{58911}$ and F_{891011} components are nonzero, but these are vanishing in the case of [15]. For 4-forms with upper indices, there are nonzero components F^{45m10}, F^{4m1011} , and F^{5m1011} ($m = 6, 7, 8, 9$) and nonzero components with the same indices of lower 4-forms obtained previously except the following components F^{4mnp}, F^{5mnp} , and F^{mnp11} ($m, n, p = 6, 7, 8, 9$) that are vanishing.

References

- [1] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP **0810**, 091 (2008). [arXiv:0806.1218 [hep-th]].
- [2] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].
- [3] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200]; E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150]; S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].
- [4] C. Ahn and J. Paeng, “Three-dimensional SCFTs, supersymmetric domain wall and renormalization group flow,” Nucl. Phys. B **595**, 119 (2001) [arXiv:hep-th/0008065].
- [5] C. Ahn and K. Woo, “Supersymmetric domain wall and RG flow from 4-dimensional gauged N = 8 supergravity,” Nucl. Phys. B **599**, 83 (2001). [arXiv:hep-th/0011121]; C. Ahn and S. J. Rey, “More CFTs and RG flows from deforming M2/M5-brane horizon,” Nucl. Phys. B **572**, 188 (2000). [arXiv:hep-th/9911199].
- [6] C. Ahn and T. Itoh, “An N = 1 supersymmetric G(2)-invariant flow in M-theory,” Nucl. Phys. B **627**, 45 (2002). [arXiv:hep-th/0112010].
- [7] R. Corrado, K. Pilch and N. P. Warner, “An N = 2 supersymmetric membrane flow,” Nucl. Phys. B **629**, 74 (2002) [arXiv:hep-th/0107220].
- [8] C. Ahn, “Holographic Supergravity Dual to Three Dimensional N=2 Gauge Theory,” JHEP **0808**, 083 (2008). [arXiv:0806.1420 [hep-th]].
- [9] M. Benna, I. Klebanov, T. Klose and M. Smedback, “Superconformal Chern-Simons Theories and AdS_4/CFT_3 Correspondence,” JHEP **0809**, 072 (2008). [arXiv:0806.1519 [hep-th]]; I. Klebanov, T. Klose and A. Murugan, “ AdS_4/CFT_3 – Squashed, Stretched and Warped,” JHEP **0903**, 140 (2009). [arXiv:0809.3773 [hep-th]]; I. R. Klebanov, S. S. Pufu and F. D. Rocha, “The Squashed, Stretched, and Warped Gets Perturbed,” JHEP **0906**, 019 (2009) [arXiv:0904.1009 [hep-th]].

- [10] C. Ahn, “Towards Holographic Gravity Dual of N=1 Superconformal Chern-Simons Gauge Theory,” JHEP **0807**, 101 (2008). [arXiv:0806.4807 [hep-th]]; C. Ahn, “Comments on Holographic Gravity Dual of N=6 Superconformal Chern-Simons Gauge Theory,” JHEP **0903**, 107 (2009). [arXiv:0812.4363 [hep-th]].
- [11] N. Bobev, N. Halmagyi, K. Pilch and N. P. Warner, “Holographic, N=1 Supersymmetric RG Flows on M2 Branes,” JHEP **0909**, 043 (2009) [arXiv:0901.2736 [hep-th]].
- [12] C. Ahn and K. Woo, “N=8 Gauged Supergravity Theory and N=6 Superconformal Chern-Simons Matter Theory,” arXiv:0901.4666 [hep-th]; C. Ahn and K. Woo, “Are There Any New Vacua of Gauged N=8 Supergravity in Four Dimensions?,” arXiv:0904.2105 [hep-th]; C. Ahn, “The Gauge Dual of Gauged N=8 Supergravity Theory,” arXiv:0905.3943 [hep-th].
- [13] C. Ahn and K. Woo, “Perturbing Around A Warped Product Of AdS_4 and Seven-Ellipsoid,” JHEP **0908**, 065 (2009) [arXiv:0907.0969 [hep-th]].
- [14] C. Ahn and K. Woo, “The Gauge Dual of A Warped Product of AdS_4 and A Squashed and Stretched Seven-Manifold,” arXiv:0908.2546 [hep-th].
- [15] C. Ahn, “New N=2 Supersymmetric Membrane Flow In Eleven-Dimensional Supergravity,” arXiv:0909.3745 [hep-th].
- [16] M. Cvetič, H. Lu, D. N. Page and C. N. Pope, “New Einstein-Sasaki spaces in five and higher dimensions,” Phys. Rev. Lett. **95**, 071101 (2005) [arXiv:hep-th/0504225].
- [17] D. Martelli and J. Sparks, “Toric Sasaki-Einstein metrics on $S^2 \times S^3$,” Phys. Lett. B **621**, 208 (2005) [arXiv:hep-th/0505027].
- [18] C. Ahn and J. F. Vazquez-Poritz, “Marginal deformations with $U(1)^3$ global symmetry,” JHEP **0507**, 032 (2005) [arXiv:hep-th/0505168].
- [19] C. Ahn and T. Itoh, “The 11-dimensional metric for AdS/CFT RG flows with common $SU(3)$ invariance,” Nucl. Phys. B **646**, 257 (2002).
- [20] E. Cremmer, B. Julia and J. Scherk, “Supergravity theory in 11 dimensions,” Phys. Lett. B **76**, 409 (1978).
- [21] M. J. Duff, B. E. W. Nilsson and C. N. Pope, “Kaluza-Klein Supergravity,” Phys. Rept. **130**, 1 (1986).

- [22] P. G. O. Freund and M. A. Rubin, “Dynamics Of Dimensional Reduction,” Phys. Lett. B **97**, 233 (1980).
- [23] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, “A new infinite class of Sasaki-Einstein manifolds,” Adv. Theor. Math. Phys. **8**, 987 (2006) [arXiv:hep-th/0403038].
- [24] S. Franco, I. R. Klebanov and D. Rodriguez-Gomez, “M2-branes on Orbifolds of the Cone over $Q^{1,1,1}$,” JHEP **0908**, 033 (2009) [arXiv:0903.3231 [hep-th]].